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An energy-based damage model of geomaterials—II. Deduction of damage evolution laws

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Abstract

This paper deals with the establishment of anisotropic conjugate force-based damage evolution laws in the framework of Rice's (1971) 'normality structure'. Attention is particularly focused on the second-order damage tensors based on the fabric tensor, which represents preexisting Griffith microcracks in solids during small deformation, isothermal and time-dependent processes. The principal results include the deduced damage surfaces, potentials and kinetic equations for the basic internal variables, damage vectors and damage tensor. For the damage tensor it is shown that the deduced damage surfaces in affinity space have a parallel relation with the Tresca and Mises yield conditions in stress space and that the damage characteristic tensor **J** can be determined uniquely by the current damage tensor without resorting to microscopic parameters or unknown empirical coefficients.

1. Introduction

In anisotropic damage models with second-order damage tensors, the damage evolution laws have been the most elusive parts due to their complex tensorial and high-degree nonlinear properties. Of various forms, the damage evolution laws in affinity space based on the linear irreversible thermodynamics are the ones most often used,

$$\dot{\boldsymbol{\Omega}} = \mathbf{J}:\mathbf{Y} \tag{1}$$

where $\boldsymbol{\Omega}$ is the second-order damage tensor; $\mathbf{Y}$ is the thermodynamic force conjugate to the damage tensor; and $\mathbf{J}$ is the damage characteristic tensor of rank four, see e.g., Chow and Lu (1989). The damage evolution law defined by eqn (1) is also termed the phenomenological equation in irreversible thermodynamics, see e.g. Malvern (1969). In this paper, the essential problem is to identify

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the conditions of the phenomenological eqn (1) and the specific form of the damage characteristic tensor **J**.

The essential problem cannot be solved in phenomenological damage models, where the linear rule eqn (1) is taken as a prerequisite rather than a conclusion. Even if the linear rule really holds true, irreversible thermodynamics or current available experimental data is not enough to determine the damage characteristic tensors **J** uniquely. Chow and Lu (1989) summarized various **J** tensors, e.g., from Chaboche, Lee, Murakami–Ohno, Sidoroff–Cordebois, etc., which generally contained one or two unknown empirical coefficients and had oversimplified tensorial forms. The reason is simple. From a purely tensorial viewpoint, two coefficients can only describe an isotropic tensor of rank four; on the other hand, the introduction of more unknown coefficients makes a phenomenological model less practicable.

Micromechanical methods appear to be feasible ways to remove the uncertainties in anisotropic models. Deducing a conjugate force-based damage evolution law from a stress-based microscopic model is, however, very difficult if not impossible, especially for a second-order damage tensor model. Some micromechanical damage models have been proposed to bridge the gap, e.g., Krajcinovic et al. (1991), Krajcinovic and Fonseka (1981) and Ilankamban and Krajcinovic (1987). However, these were not damage tensor models and still belonged to phenomenological models, because their basic relations were postulated. In these models, essentially only the geometric features of the microscopic model were incorporated into constitutive models. On the other hand, some 'complete' micromechanical models, in which macroscopic behaviour was fully determined by the microscopic defect models, were proposed and possessed the least ambiguity, see e.g., Ju and Lee (1991), Kachanov (1982), Dragon (1985), etc. Unfortunately, their final macroscopic constitutive equations were stress-based.

In this paper, the essential problem is solved in the framework of Rice's (1971) 'normality structure'. The basic internal variables $\boldsymbol{\xi}$ are the numerous vector form variables, and each of them corresponds to one Griffith microcrack weakening the solid. The damage tensor $\boldsymbol{\Omega}$ is taken as the average measure for $\boldsymbol{\xi}$. Damage vectors $\boldsymbol{\zeta}$ are introduced as the intermediate average measures between $\boldsymbol{\xi}$ and $\boldsymbol{\Omega}$. To focus on the essential problems, only small deformation, isothermal and time-independent processes are considered. The damage surfaces, potentials and kinetic equations on the three levels, $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$ and $\boldsymbol{\Omega}$, have been deduced exclusively in the affinity space without considering the specific form of the Gibbs energy or free energy. The interaction of the microcracks is taken into account implicitly since the normality structure is within the framework of the self-consistent method.

The starting point of this deduction is the kinetic equation of one basic internal variable, as defined in eqn (14). It represents such a class of materials to which the influence of the macroscopic stress/strain appears only through the conjugate forces. Since one basic internal variable corresponds to one microcrack, its kinetic equation should be consistent with the cracking criterion in fracture mechanics. In Section 5, some numerical tests have been done to show the behaviour of the kinetic equation and make a comparison between the two criteria.

2. Damage definitions

Consider a material sample of size V^0 weakened by n microcracks. The microstructure of the sample can be characterized using the following set of basic internal variables:

$$\boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots, \xi_n\}, \quad \xi_\alpha = \{\mathbf{n}^\alpha, r_\alpha\} \tag{2}$$

where $\mathbf{n}^\alpha$ and r_α are the normal vector and radius, respectively, of α-th microcrack.

To much reduce the number of the basic internal variables, $m(m \ll n)$ damage vector ζ_μ are introduced as the average measure of $\boldsymbol{\xi}$

$$\boldsymbol{\zeta} = \{\zeta_1, \zeta_2, \ldots, \zeta_m\}, \quad \zeta_\mu = \{\mathbf{n}^\mu, \omega_\mu\} \tag{3}$$

where the μ-th damage vector $\zeta_\mu = \{\mathbf{n}^\mu, \omega_\mu\}$ is the average measure of one group of n_μ roughly parallel microcracks among all microcracks. Without a loss of generality, μ can be dropped in the following average expression

$$\mathbf{n} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{n}^i, \quad \omega = \frac{1}{V^0}\sum_{i=1}^{n} r_i^3 \tag{4}$$

The second-order damage tensor $\boldsymbol{\Omega}_{ij}$ is introduced just as the average measure of the damage vectors

$$\boldsymbol{\Omega} = \sum_{\mu=1}^{m} \omega_\mu \mathbf{n}^\mu \mathbf{n}^\mu \tag{5}$$

The damage variable ω takes the form of the third-order moment of microcrack radii, as shown in eqn (4b). The damage tensor is just the second-order fabric tensor:

$$\boldsymbol{\Omega} = \frac{1}{V^0}\sum_{\alpha=1}^{n} r_\alpha^3 \mathbf{n}^\alpha \mathbf{n}^\alpha \tag{6}$$

if the cracks in each group, see eqn (4), are parallel exactly with each other.

3. Normality structure

The framework illustrated here is generally the same as Rice's (1971) original 'normality structure'. Formulation is extended to have a unitary form for both time-dependent and time-independent behaviour. Here only the small deformation and isothermal processes have been considered, rather than the original finite strain and thermo-elasticity. It is noted that one internal variable, ξ_α or ζ_μ should be understood as a vector (originally scalar) in this paper. Introduce the specific free energy ϕ and its Legendre transform ψ with respect to strain $\boldsymbol{\varepsilon} = \varepsilon_{ij}$

$$\phi = \phi(\boldsymbol{\varepsilon}, \boldsymbol{\xi}) = u - \theta\eta, \quad \psi = \psi(\boldsymbol{\sigma}, \boldsymbol{\xi}) = \boldsymbol{\varepsilon} : \frac{\partial \phi}{\partial \boldsymbol{\varepsilon}} - \phi \tag{7}$$

where $\boldsymbol{\sigma} = \sigma_{ij}$ is the stress tensor; u specific internal energy, η specific entropy; θ temperature; and ψ the complementary energy also termed Gibbs energy. When neighbouring constrained equilibrium states corresponding to different sets of internal variables are considered, the first law of thermodynamics takes the form

$$\boldsymbol{\varepsilon}:\dot{\boldsymbol{\sigma}}+\frac{1}{V^0}f_\alpha\dot{\xi}_\alpha=\dot{\psi} \tag{8}$$

which defines the thermodynamic forces $f_1, f_2, \ldots, f_n$ (collectively $\mathbf{f}$) acting on the internal variables,

$$f_\alpha = V^0\frac{\partial\psi(\boldsymbol{\sigma},\boldsymbol{\xi})}{\partial\xi_\alpha} = -V^0\frac{\partial\phi(\boldsymbol{\varepsilon},\boldsymbol{\xi})}{\partial\xi_\alpha} \tag{9}$$

and leads to

$$\boldsymbol{\varepsilon}=\frac{\partial\psi(\boldsymbol{\sigma},\boldsymbol{\xi})}{\partial\boldsymbol{\sigma}},\quad \boldsymbol{\sigma}=\frac{\partial\phi(\boldsymbol{\varepsilon},\boldsymbol{\xi})}{\partial\boldsymbol{\varepsilon}} \tag{10}$$

The second law of thermodynamics further requires $f_\alpha\dot{\xi}_\alpha \geqslant 0$. Since $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(\boldsymbol{\sigma},\boldsymbol{\xi})$ from eqn (10), the strain rate can be expressed as

$$\dot{\boldsymbol{\varepsilon}}=\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}+\dot{\boldsymbol{\varepsilon}}^{\mathrm{d}},\quad \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}=\frac{\partial\boldsymbol{\varepsilon}(\boldsymbol{\sigma},\boldsymbol{\xi})}{\partial\boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}},\quad \dot{\boldsymbol{\varepsilon}}^{\mathrm{d}}=\frac{\partial\boldsymbol{\varepsilon}(\boldsymbol{\sigma},\boldsymbol{\xi})}{\partial\xi_\alpha}\dot{\xi}_\alpha \tag{11}$$

where $\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}$ is the elastic strain rate due to the change of stress at the fixed internal variables, and can be expressed by using eqn (10)

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}=\frac{\partial^2\psi}{\partial\boldsymbol{\sigma}\,\partial\boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}} \tag{12}$$

and $\dot{\boldsymbol{\varepsilon}}^{\mathrm{d}}$ is the inelastic strain rate[1] due to the change of the internal variables and can be expressed using eqns (10) and (9)

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{d}}=\frac{1}{V^0}\frac{\partial f_\alpha(\boldsymbol{\sigma},\boldsymbol{\xi})}{\partial\boldsymbol{\sigma}}\dot{\xi}_\alpha \tag{13}$$

In the normality structure, the kinetic equations of internal variables are required to take the form

$$\dot{\xi}_\alpha=\dot{\xi}_\alpha(f_\alpha,\boldsymbol{\xi}),\quad (\alpha=1,2,\ldots,n) \tag{14}$$

Then, Q is a point function if set

$$Q=Q(\mathbf{f},\boldsymbol{\xi})=\frac{1}{V^0}\int_0^{\mathbf{f}}\dot{\xi}_\alpha(\mathbf{f},\boldsymbol{\xi})\,\mathrm{d}f_\alpha \tag{15}$$

Therefore, the kinetic equation of internal variables can be recast as

$$\dot{\xi}_\alpha=V^0\frac{\partial Q(\mathbf{f},\boldsymbol{\xi})}{\partial f_\alpha} \tag{16}$$

The function Q is just the potential function of inelastic strain. Note that $Q=Q(\mathbf{f},\boldsymbol{\xi})=Q(\boldsymbol{\sigma},\boldsymbol{\xi})$ since $\mathbf{f}=\mathbf{f}(\boldsymbol{\sigma},\boldsymbol{\xi})$. Thus, in view of eqns (13) and (16),

[1] The framework in this section is suitable to accommodate either plasticity or damage.

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{d}} = \frac{\partial Q(\boldsymbol{\sigma}, \boldsymbol{\xi})}{\partial \boldsymbol{\sigma}} \tag{17}$$

Since ξ_α corresponds to one microcrack, the total number n of $\boldsymbol{\xi}$ is very large. One set of much reduced internal variables $\boldsymbol{\zeta}$ can be introduced as the average measures of $\boldsymbol{\xi}$

$$\zeta_\mu = \zeta_\mu(\xi_1, \xi_2, \ldots, \xi_n; V^0) \quad (\mu = 1, 2, \ldots, m \ll n) \tag{18}$$

where V^0 indicates averaging over the volume. Formulation in terms of the average internal variables $\boldsymbol{\zeta}$ is parallel to that in terms of $\boldsymbol{\xi}$

$$\begin{aligned} &\boldsymbol{\varepsilon} = \frac{\partial \psi(\boldsymbol{\sigma}, \boldsymbol{\zeta})}{\partial \boldsymbol{\sigma}}, \quad g_\mu = \frac{\partial \psi(\boldsymbol{\sigma}, \boldsymbol{\zeta})}{\partial \zeta_\mu}, \quad \boldsymbol{\varepsilon}^{\mathrm{d}} = \frac{\partial g_\mu(\boldsymbol{\sigma}, \boldsymbol{\zeta})}{\partial \boldsymbol{\sigma}} \dot{\zeta}_\mu \\ &\dot{\boldsymbol{\zeta}}_\mu = \frac{\partial Q(\mathbf{g}, \boldsymbol{\zeta})}{\partial g_\mu}, \quad \boldsymbol{\varepsilon}^{\mathrm{d}} = \frac{\partial Q(\boldsymbol{\sigma}, \boldsymbol{\zeta})}{\partial \boldsymbol{\sigma}}, \quad Q(\mathbf{g}, \boldsymbol{\zeta}) = \int_0^{\mathbf{g}} \dot{\zeta}_\mu(\mathbf{g}, \boldsymbol{\zeta}) \, \mathrm{d}g_\mu \end{aligned} \tag{19}$$

where $g_1, g_2, \ldots, g_m$ (collectively $\mathbf{g}$) are the thermodynamic forces acting on the internal variable $\boldsymbol{\zeta}$. The equivalence for the average measure $\boldsymbol{\zeta}$ to describe the thermodynamic system characterized by its basic internal variable, $\boldsymbol{\xi}$, is achieved by requiring the equality for all $\delta\boldsymbol{\xi}$

$$g_\mu \delta \zeta_\mu = \frac{1}{V^0} f_\alpha \delta \xi_\alpha \Rightarrow Q(\mathbf{g}, \boldsymbol{\zeta}) = Q(\mathbf{f}, \boldsymbol{\xi}) \tag{20}$$

In other words, the equivalence is achieved by recasting $Q(\mathbf{f}, \boldsymbol{\xi})$ as $Q(\mathbf{g}, \boldsymbol{\zeta})$.

The kinetic equations eqn (14) is generally suitable for time-dependent behaviour. For time-independent behaviour, the existence of the potential Q in eqn (15) is questionable, because not all internal variables are certainly active under a loading. In order to extend the formulation to take into account time-independent behaviour, the damage (yield) surfaces $F_1, F_2, \ldots, F_n$ (collectively $\mathbf{F}$) for basic internal variables $\boldsymbol{\xi}$ are introduced, and the kinetic equation eqn (14) can be recast as

$$\dot{\xi}_\alpha = \dot{\xi}_\alpha(f_\alpha, \boldsymbol{\xi}) = \dot{\lambda}_\alpha \mathbf{K}^\alpha(f_\alpha, \boldsymbol{\xi}), \quad (\alpha = 1, 2, \ldots, n) \tag{21}$$

where

$$\dot{\lambda}_\alpha = \begin{cases} > 0 & \text{if } F_\alpha = 0 \quad \text{and} \quad \dfrac{\partial F_\alpha}{\partial f_\alpha} \dot{f}_\alpha > 0 \quad \text{(no summation for } \alpha) \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

where $\dot{\lambda}_\alpha$ can be determined by the consistency condition $\dot{F}_\alpha = 0$ if $\dot{\lambda}_\alpha > 0$. Thus, the potential function Q in eqn (15) still exists in form but should be rewritten as $Q(\mathbf{f}, \boldsymbol{\xi}, \mathbf{F})$ to reflect its dependence on the damage surfaces $\mathbf{F}$. Similarly, there also exist damage (yield) surfaces $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_m$ (collectively $\mathscr{F}$) for the condensed internal variables $\boldsymbol{\zeta}$ and the corresponding potential function $Q(\mathbf{g}, \boldsymbol{\zeta}, \mathscr{F})$.

In short, the first step in the normality structure is to establish the kinetic equations and damage surfaces of the basic internal variables $\boldsymbol{\xi}$. The main task in pursuing the kinetic equations of the condensed internal variable $\boldsymbol{\zeta}$ from that of $\boldsymbol{\xi}$, in reality, is the transformation $(\mathbf{f}, \boldsymbol{\xi}, \mathbf{F}) \Rightarrow (\mathbf{g}, \boldsymbol{\zeta}, \mathscr{F})$. The crux in the transformation is the establishment of the relation, $\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{\zeta})$ which is the inverse

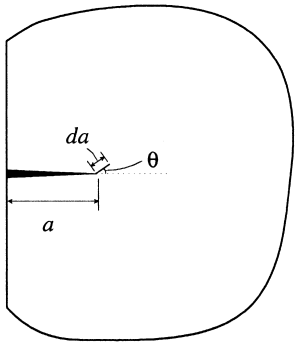

Fig. 1. Schematic crack propagation.

function of eqn (18). The inverse function cannot be determined uniquely by eqn (18) since $\boldsymbol{\zeta}$, as an average measure of $\boldsymbol{\xi}$, is determined uniquely by $\boldsymbol{\xi}$ but not conversely. It is the uncertainty that reflects mathematically the fact that $\boldsymbol{\zeta}$ can only furnish an approximate description for a thermodynamic system characterized by $\boldsymbol{\xi}$.

In this paper, we deal with the uncertainty by replacing $\boldsymbol{\xi}$ with $\hat{\boldsymbol{\xi}}$. Here, $\hat{\boldsymbol{\xi}}$ is one set of fictitious basic internal variables and subject to the conditions: (1) $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\xi}) = \boldsymbol{\zeta}(\hat{\boldsymbol{\xi}})$; (2) $\hat{\boldsymbol{\xi}}$ should be determined uniquely by $\boldsymbol{\zeta}$.

The preceding formulation takes crack interactions into account implicitly. As pointed out by Krajcinovic et al. (1991), the normality structure is exactly within the framework of the self-consistent method.

4. Kinetic equations of basic internal variables

The kinetic equations of basic internal variables, $\dot{\xi}_\alpha(f_\alpha, \boldsymbol{\xi})$, are the cornerstones of the normality structure. These equations comprise the damage potential Q, which is the basis for the kinetic equations of damage variables $\boldsymbol{\zeta}$ and $\boldsymbol{\Omega}$. Since ξ_α corresponds to one microcrack, $\dot{\xi}_\alpha(f_\alpha, \boldsymbol{\xi})$ must be compatible with cracking criteria in fracture mechanics.

4.1. *Cracking criteria*

Of several types of cracking criteria in linear fracture mechanics, we shall consider the energy principle. For a specimen in Fig. 1 subjected to certain external forces, the energy principle (see e.g. Broek, 1987) in accordance with Griffith's criterion, postulates that cracking initiates if

$$G = R \tag{23}$$

where R is the crack resistance or R-curve, and G is the energy release rate.

$$G = \frac{\mathrm{d}(W-U)}{\mathrm{d}a} \tag{24}$$

where U is the elastic energy contained in the specimen, and W is the work performed by the external force.

In general, the crack propagates not along the original plane but with kinking, see Fig. 1. The well-known criteria to determine the kinking angle θ include: (1) $\sigma(\theta)_{\max}$ theory by Erdogan and Sih (1963) that crack extension starts in a plane normal to the direction of maximum circumferential stress $\sigma(\theta)_{\max}$ near the crack tip; (2) $S(\theta)_{\min}$ theory by Sih (1973) that crack extension occurs in the direction along which strain energy density $S(\theta)$ possesses a minimum value; (3) $G(\theta)_{\max}$ theory by Hussain et al. (1974) that crack extension starts in the direction along which energy release rate $G(\theta)$ possesses a maximum value.

$G(\theta)_{\max}$ theory is adopted in this section, because it is an energy principle and consistent with the principle of maximum dissipation in thermodynamics. Note that $(W-U)$ in eqn (24) is the dissipated energy for brittle materials. Therefore, $G(\theta)_{\max}$ theory implies that, for a given crack increment da, the system tends to dissipate maximum energy by cracking at a specific kinking angle θ.

4.2. *Rotation rate*

Consider a specimen of volume V^0 with n microcracks. The dissipated energy of the α-th microcrack during its propagation is

$$\mathrm{d}(W-U) = f_\alpha \,\mathrm{d}\xi_\alpha \quad \text{(no summation for } \alpha) \tag{25}$$

The following discussions all focus on the α-th microcrack. Thus, we can drop α in the section without a loss of generality. The conjugate force f defined in eqn (9) should possess a vector form

$$f = V^0 \frac{\partial \psi}{\partial \xi} = \{\mathbf{f}_\mathrm{n}, f_\mathrm{r}\} = V^0 \left\{\frac{\partial \psi}{\partial \mathbf{n}}, \frac{\partial \psi}{\partial r}\right\} = -V^0 \left\{\frac{\partial \phi}{\partial \mathbf{n}}, \frac{\partial \phi}{\partial r}\right\} \tag{26}$$

Thus, the dissipated energy can be recast as

$$\mathrm{d}(W-U) = f\,\mathrm{d}\xi = \mathbf{f}_\mathrm{n} \cdot \mathrm{d}\mathbf{n} + f_\mathrm{r}\,\mathrm{d}r \tag{27}$$

Note that the crack length increment da in eqn (24) should be replaced with the crack area increment dA in a general 3-D case. The area vector of the penny crack can be defined as

$$\mathbf{A} = \pi r^2 \mathbf{n} \Rightarrow \mathrm{d}\mathbf{A} = \pi r^2\,\mathrm{d}\mathbf{n} + 2\pi r \mathbf{n}\,\mathrm{d}r \tag{28}$$

and note that

$$\mathbf{n}\cdot\mathbf{n} = 1 \Rightarrow \mathbf{n}\cdot\dot{\mathbf{n}} = 0 \tag{29}$$

Thus, the area increment of the penny crack is obtained as

$$\mathrm{d}A = |\mathrm{d}\mathbf{A}| = \sqrt{\mathrm{d}\mathbf{A}\cdot\mathrm{d}\mathbf{A}} = 2\pi r\sqrt{1+\mathbf{z}\cdot\mathbf{z}}\,\mathrm{d}r \tag{30}$$

where

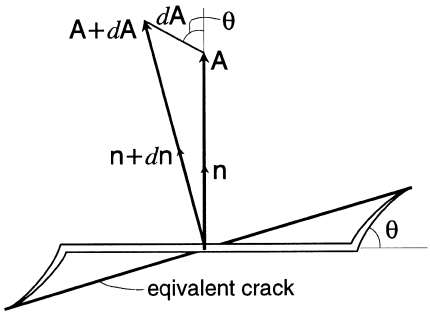

Fig. 2. Schematic kinking crack equivalence.

$$\mathbf{z} = \frac{r}{2}\frac{\dot{\mathbf{n}}}{\dot{r}} \tag{31}$$

The kinking angle θ of the crack may be defined in an average sense as

$$\cos\theta = \mathbf{n}\cdot\frac{\mathrm{d}\mathbf{A}}{|\mathrm{d}\mathbf{A}|} = \frac{1}{\sqrt{1+\mathbf{z}\cdot\mathbf{z}}} \Rightarrow \tan\theta = |\mathbf{z}| = \sqrt{\mathbf{z}\cdot\mathbf{z}} \tag{32}$$

which is schematically illustrated in Fig. 2. Using eqns (27) and (30), the energy release rate of the α-th microcrack is found to be

$$G = \frac{\mathrm{d}(W-U)}{\mathrm{d}A} = \frac{2\mathbf{f}_{\mathrm{n}}\cdot\mathbf{z}+f_{\mathrm{r}}r}{2\pi r^2\sqrt{1+\mathbf{z}\cdot\mathbf{z}}} \tag{33}$$

Since $\mathbf{z}$ determines the kinking angle θ uniquely, the $G(\theta)_{\max}$ theory requires that the actual $\mathbf{z}$ should maximize the energy release rate G. Note that due to eqn (29) the components of $\mathbf{z}$ are not mutually independent:

$$\mathbf{n}\cdot\mathbf{z} = 0 \tag{34}$$

Thus, the maximization problem corresponds to such a Lagrangian extreme-value problem,

$$\mathscr{L} = G+\lambda\mathbf{n}\cdot\mathbf{z}, \quad \frac{\partial\mathscr{L}}{\partial\mathbf{z}} = 0 \tag{35}$$

Omitting the detailed deduction (see Yang, 1996), the rotation rate is obtained as

$$\mathbf{z} = \frac{2}{f_{\mathrm{r}}r}(\mathbf{f}_{\mathrm{n}}-\mathbf{f}_{\mathrm{nn}}\cdot\mathbf{n}), \quad \text{or } \dot{\mathbf{n}} = \frac{4(\mathbf{f}_{\mathrm{n}}-\mathbf{f}_{\mathrm{n}}\cdot\mathbf{nn})}{f_{\mathrm{r}}r^2}\dot{r} \tag{36}$$

4.3. *Kinetic equations*

Equation (36) can be recast as an equivalent form

$$\{\dot{\mathbf{n}},\dot{r}\} = \dot{\lambda}\left\{(\mathbf{I}-\mathbf{nn})\cdot\mathbf{f}_{\mathrm{n}}, \left(\frac{r}{2}\right)^2 f_{\mathrm{r}}\right\} \tag{37}$$

or

$$\dot{\xi} = \dot{\lambda}\mathbf{K}, \quad \mathbf{K} = \left\{(\mathbf{I}-\mathbf{nn})\cdot\mathbf{f}_{\mathrm{n}}, \left(\frac{r}{2}\right)^2 f_{\mathrm{r}}\right\} \tag{38}$$

which is exactly the required kinetic equation eqn (21). In accordance with eqn (23), the local damage surface can take the form

$$F = G - R \tag{39}$$

where energy release rate G is obtained by substituting $\mathbf{z}$ into eqn (33)

$$G = \frac{1}{\pi r^2}\sqrt{\mathbf{f}_{\mathrm{n}}\cdot\mathbf{f}_{\mathrm{n}} - (\mathbf{f}_{\mathrm{n}}\cdot\mathbf{n})^2 + \left(\frac{f_{\mathrm{r}} r}{2}\right)^2} \tag{40}$$

or an equivalent form

$$G = \sqrt{G_{\mathrm{n}}^2 + G_{\mathrm{t}}^2}, \quad \text{where } G_{\mathrm{n}} = \frac{1}{\pi r^2}\sqrt{\mathbf{f}_{\mathrm{n}}\cdot\mathbf{f}_{\mathrm{n}} - (\mathbf{f}_{\mathrm{n}}\cdot\mathbf{n})^2}, \quad G_{\mathrm{t}} = \frac{f_{\mathrm{r}}}{2\pi r} \tag{41}$$

In view of eqns (32) and (36), it is easy to prove the relation

$$\tan\theta = \frac{G_{\mathrm{n}}}{G_{\mathrm{t}}} \tag{42}$$

If $G_{\mathrm{n}} = 0$, then $\theta = 0$, which means no kinking. Thus, G_{n} is the drive force behind the kinking of the crack. In brief, the energy release rate G can be decomposed into two components: (1) normal component G_{n} to drive the crack to rotate without propagation; (2) tangent component G_{t} to drive the crack to propagate without rotation.

The R curve can be expressed as the function of crack growth Δa, namely, $R = R(\Delta a)$, see Krafft et al. (1961). At a macroscopic level, crack resistance R is proposed as $R = R(H)$, where H is the history recording parameter. Therefore, H is consistent with Δa at a microscopic level. It thus follows that $\mathrm{d}H$ can be determined in an averaging sense:

$$\begin{aligned}\mathrm{d}H = \mathrm{d}r &= \frac{\mathrm{d}A}{2\pi r} = \sqrt{1+\mathbf{z}\cdot\mathbf{z}}\,\mathrm{d}r \\ &= \frac{r}{2}\sqrt{\mathbf{f}_{\mathrm{n}}\cdot\mathbf{f}_{\mathrm{n}} - (\mathbf{f}_{\mathrm{n}}\cdot\mathbf{n})^2 + \left(\frac{f_{\mathrm{r}} r}{2}\right)^2}\,\mathrm{d}\lambda = \frac{\pi r^3}{2} G\,\mathrm{d}\lambda\end{aligned} \tag{43}$$

Thus, the threshold R can be defined as

$$R = R(H), \quad \dot{H} = \dot{\lambda} h(f, \xi), \quad h = \frac{\pi r^3}{2} G \tag{44}$$

5. Numerical tests on cracking

The kinetic equation of the basic internal variable, eqn (38), as a specific form of eqn (14), is the cornerstone of the normality structure. The subsequent deduction makes sense only if eqn

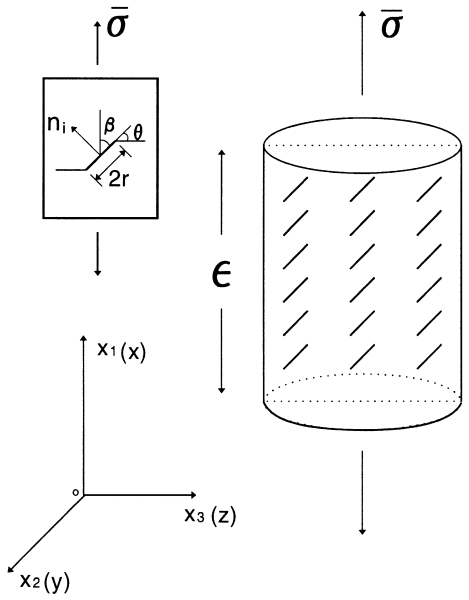

Fig. 3. A specimen with one set crack array under tension.

(38) can predict crack propagation with satisfactory accuracy since one basic internal variable corresponds to one crack. Firstly, an explicit Gibbs energy (or complementary energy) function ψ in stress space should be established to determine the conjugate force.

Consider a perfectly brittle solid of volume V^0 weakened by n parallel penny-shaped cracks with the same radii r, i.e., $\xi_\alpha = \{\mathbf{n}, r\}$, as shown in Fig. 3. Assuming self-similar growth of penny-shaped cracks, the Gibbs energy takes the following form, see, e.g., Krajcinovic et al. (1991) and Budiansky and O'Connel (1976)

$$\psi = \psi^0 + \rho r^3 (A\sigma^2 + B\tau^2) \tag{45}$$

where ψ^0 is the Gibbs energy of the virgin material under the same stress state; $\rho = (n/V^0)$ is the volumetric density of cracks; σ and τ are normal and shear components, respectively, of stress tensor σ_{ij} on the crack surface with the normal vector n_i

$$\sigma = n_i \sigma_{ij} n_j, \quad \tau = \sqrt{\sigma_i \sigma_i - \sigma^2}, \quad \sigma_i = \sigma_{ij} n_j; \tag{46}$$

A, B are associate with the crack shape and the elastic constants of effective medium around the crack. If the crack is penny-shaped and crack interaction is neglected, an explicit form of A, B is obtained

$$A = \frac{8}{3E_0}(1 - v_0^2) \quad B = \frac{2}{2 - v_0} A \tag{47}$$

where E_0, v_0 are elastic constants of intrinsic elastic media. Note that the explicit A, B is helpful to

obtain analytic results, but it goes beyond the framework of the self-consistent method at this point. Then the conjugate forces can be determined with eqn (26)[2]

$$(f_{\mathrm{n}})_i = 2r^3[(2(A-B)\sigma_i\sigma + B\sigma_{ij}\sigma_j], \quad f_{\mathrm{r}} = 3r^2(A\sigma^2 + B\tau^2) \tag{48}$$

With eqn (41), the energy release rates are

$$G_{\mathrm{n}} = \frac{2r}{\pi}\sqrt{4(A-B)^2 s_1 + 4(A-B)Bs_2 + B^2 s_3}, \quad G_{\mathrm{t}} = \frac{3r}{2\pi}(A\sigma^2 + B\tau^2) \tag{49}$$

where

$$s_1 = \sigma^2\tau^2, \quad s_2 = \sigma\sigma_i\sigma_{ij}\sigma_j - \sigma^2\sigma_k\sigma_k, \quad s_3 = \sigma_i\sigma_{im}\sigma_{mj}\sigma_j - (\sigma_k\sigma_k)^2 \tag{50}$$

which are stress invariants on the crack surface. In view of eqn (36)

$$\dot{n}_i = \frac{8}{3r}\frac{2(A-B)(\sigma_i\sigma - n_i\sigma^2) + B(\sigma_{ij}\sigma_j - n_i\sigma_k\sigma_k)}{A\sigma^2 + B\tau^2}\dot{r} \tag{51}$$

The kinking angle is determined by eqn (42),

$$\tan\theta = \frac{G_{\mathrm{n}}}{G_{\mathrm{t}}} = \frac{4}{3}\frac{\sqrt{4(A-B)^2 s_1 + 4(A-B)Bs_2 + B^2 s_3}}{A\sigma^2 + B\tau^2} \tag{52}$$

5.1. *Kinking angle*

If the specimen in Fig. 3 is subjected to a uniform tensile stress $\bar{\sigma}$, the stresses defined in eqn (46) become

$$\sigma_i = \{\bar{\sigma}\sin\beta, 0, 0\}, \quad \sigma = \bar{\sigma}\sin^2\beta, \quad \tau = \bar{\sigma}\sin\beta\cos\beta \tag{53}$$

and with eqn (50)

$$s_1 = \bar{\sigma}^4\sin^6\beta\cos^2\beta, \quad s_2 = \bar{\sigma}^4\sin^4\beta\cos^2\beta, \quad s_3 = \bar{\sigma}^4\sin^2\beta\cos^2\beta \tag{54}$$

By using eqns (49) and (47)

$$G_{\mathrm{n}} = \frac{2r\bar{\sigma}^2 B\sin\beta\cos\beta}{\pi}(1 - v_0\sin^2\beta), \quad G_{\mathrm{t}} = \frac{3r\bar{\sigma}^2 B\sin^2\beta}{4\pi}(2 - v_0\sin^2\beta) \tag{55}$$

Therefore, the kinking angle θ is obtained as

$$\tan\theta = \frac{G_{\mathrm{n}}}{G_{\mathrm{t}}} = \frac{8}{3}\frac{1 - v_0\sin^2\beta}{2 - v_0\sin^2\beta}\cot\beta \tag{56}$$

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[2] Note that the conjugate force f corresponds to one single crack. Thus, $\mathbf{n}$, r in eqn (45) should be considered as the average measures of $\mathbf{n}^{\alpha}$, r_{α} during the derivation.

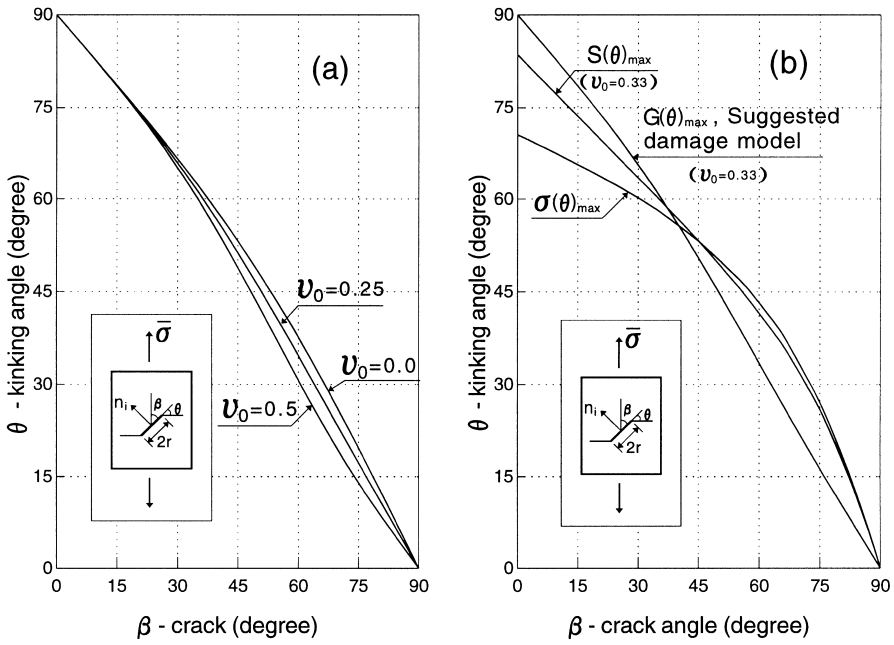

Fig. 4. Crack angle vs kinking angle in tension: (a) influences of Poisson's ratio; (b) comparison with $\sigma(\theta)_{\max}$ and $S(\theta)_{\max}$ theories.

which is illustrated in Fig. 4(a). Obviously, Poisson's ratio v_0 has little influence on the kinking angle. The same problem in Fig. 3 in a plane strain case can be solved by $\sigma(\theta)_{\max}$ theory (see Sih, 1993) as

$$\sin\theta + (3\cos\theta - 1)\cot\beta = 0 \tag{57}$$

or by $S(\theta)_{\max}$ theory

$$2(1-2v_0)\sin(\theta-2\beta) - 2\sin[2(\theta-\beta)] - \sin 2\theta = 0 \tag{58}$$

All the three solutions are illustrated in Fig. 4(b) with the same $v_0 = \frac{1}{3}$. Given the fact that the θ in eqn (56) originating from eqn (36) is an average kinking angle rather than the real one in eqns (57) and (58), it is quite evident that the proposed equation of rotation rate, eqn (36), can describe the microscopic kinking mechanism with sufficient accuracy.

5.2. *Uniaxial loading process*

In this subsection, a uniaxial loading process on the same configuration in Fig. 3 is done to show the stress–strain process macroscopically and crack rotation microscopically. The local damage surface defined in eqns (39) and (41) is

$$F = G - R = \sqrt{G_{\mathrm{n}}^2 + G_{\mathrm{t}}^2} - R = 0 \tag{59}$$

Substituting eqn (55) into eqn (59), the critical stress $\bar{\sigma}$ on the damage surface is

Table 1
Parameters used in uniaxial tensile test

E_0	50000.0	MPa
v_0	0.2	—
r_0	2.52	cm
ρ	1000.0	$1/m^3$
β_0	45.0	[°]

$$\bar{\sigma} = \sqrt{\frac{4\pi R}{rB\sin^2\beta[64\cot^2\beta(1-v_0\sin^2\beta)^2+9(2-v_0\sin^2\beta)^2]}} \tag{60}$$

In view of eqn (51),

$$\dot{n}_1 = \frac{16\cos^2\beta}{3r\sin\beta}\left(\frac{1-v_0\sin^2\beta}{2-v_0\sin^2\beta}\right)\dot{r}, \quad \dot{n}_2 = 0, \quad \dot{n}_3 = -\frac{16\cos\beta}{3r}\left(\frac{1-v_0\sin^2\beta}{2-v_0\sin^2\beta}\right)\dot{r} \tag{61}$$

The axial strain ε and axial stress $\bar{\sigma}$ are related by the apparent compliance C. During the uniaxial loading, the complementary energy $\frac{1}{2}C\bar{\sigma}^2$ should equalize ψ in eqn (45). Therefore,

$$\varepsilon = C\bar{\sigma}, \quad C = \frac{1}{E_0} + 2\rho r^3\sin^2\beta(A\sin^2\beta + B\cos^2\beta) \tag{62}$$

The states during the uniaxial loading is fully determined by eqns (60)–(62). The required parameters are listed in Table 1. The chosen R-curve, as shown Fig. 5(a), is from Indiana limestone (see Hoagland et al, 1973). Eight loading steps are used. The results are shown in Fig. 5. The first loading step is elastic. The mark (·) is put in the process curves to denote each loading step. As shown in Fig. 5(a), the critical point is the peak point corresponding to the maximum loading in the stress–strain curve in Fig. 5(c). Crack growth between initial and peak point is sub-critical growth. After the peak point, crack growth is unstable. The trajectory of the end points of the vector $\mathbf{r} = r\mathbf{n}$ in the propagating process is drawn in Fig. 5(b), which correctly reflects the tendency of crack rotation. As shown in Fig. 5(d), the brittle stress–strain behaviour is well described. In short, the kinetic equations of basic internal variables can describe microcracking very well, either microscopically or macroscopically.

6. Kinetic equations of damage vectors

Without a loss of generality, we focus on one set of roughly parallel cracks,

$$\boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots, \xi_n\}, \quad \xi_\alpha = \{\mathbf{n}^\alpha, r_\alpha\} \tag{63}$$

which corresponds to one damage vector $\zeta = \{\mathbf{n}, \omega\}$, where

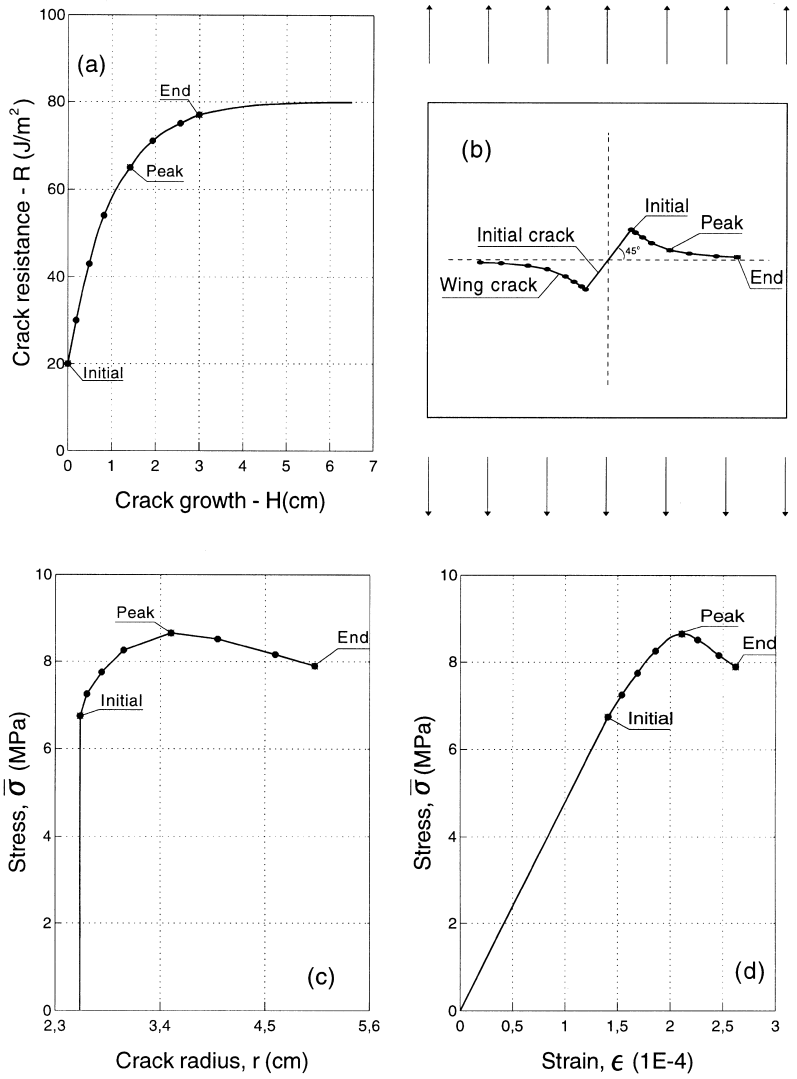

Fig. 5. (a) *R* curve; (b) damage propagating process; (c) stress vs damage growth; (d) macroscopic stress–strain relation.

$$\mathbf{n} = \frac{1}{n}\sum_{\alpha=1}^{n} \mathbf{n}^{\alpha}, \quad \omega = \frac{1}{V^0}\sum_{\alpha=1}^{n} r_{\alpha}^{3} \tag{64}$$

see also eqns (2)–(4). In terms of the basic internal variables, the potential function Q is obtained with eqns (15) and (38),

$$Q(\boldsymbol{\xi}, \mathbf{f}, \mathbf{F}) = \frac{1}{2V^0}\sum_{\alpha=1}^{n} \dot{\lambda}_{\alpha}\left[\mathbf{f}_{\mathrm{n}}^{\alpha}\cdot(\mathbf{I}-\mathbf{n}^{\alpha}\mathbf{n}^{\alpha})\cdot\mathbf{f}_{\mathrm{n}}^{\alpha} + \left(\frac{f_{\mathrm{r}}^{\alpha} r_{\alpha}}{2}\right)^{2}\right] \tag{65}$$

and the damage surface **F** with eqns (39) and (40)

$$F_\alpha = G_\alpha - R_\alpha, \quad G_\alpha = \frac{1}{\pi r_\alpha^2}\sqrt{\mathbf{f}_{\mathrm{n}}^\alpha \cdot (\mathbf{I} - \mathbf{n}^\alpha \mathbf{n}^\alpha) \cdot \mathbf{f}_{\mathrm{n}}^\alpha + \left(\frac{f_{\mathrm{r}}^\alpha r_\alpha}{2}\right)^2} \quad (\alpha = 1, 2, \ldots, n) \tag{66}$$

Now consider the transformation $(\mathbf{f}, \boldsymbol{\xi}, \mathbf{F}) \Rightarrow (\mathbf{g}, \boldsymbol{\zeta}, \mathscr{F})$. In view of eqns (26) and (64), $\mathbf{f} = \mathbf{f}(\mathbf{g})$ is achieved

$$\{\mathbf{f}_{\mathrm{n}}^\alpha, f_{\mathrm{r}}^\alpha\} = V^0 \left\{\frac{\partial \psi}{\partial \mathbf{n}^\alpha}, \frac{\partial \psi}{\partial r_\alpha}\right\} = V^0 \left\{\frac{\partial \psi}{\partial \mathbf{n}} \cdot \frac{\partial \mathbf{n}}{\partial \mathbf{n}^\alpha}, \frac{\partial \psi}{\partial \omega}\frac{\partial \omega}{\partial r_\alpha}\right\} = \left\{\frac{\mathbf{g}_{\mathrm{n}}}{\rho}, 3 r_\alpha^2 g_\omega\right\} \tag{67}$$

where $\rho = n/V^0$ is the volumetric density of the cracks; g is the thermodynamic force conjugate to the damage vector

$$g = \frac{\partial \psi}{\partial \zeta} = \left\{\frac{\partial \psi}{\partial \mathbf{n}}, \frac{\partial \psi}{\partial \omega}\right\} = \{\mathbf{g}_{\mathrm{n}}, g_\omega\} \tag{68}$$

In order to determine $\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{\zeta})$, the inverse relation of eqn (64), one set of fictitious basic internal variables $\hat{\boldsymbol{\xi}}$ is introduced to replace $\boldsymbol{\xi}$

$$\hat{\boldsymbol{\xi}} = \{\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_n\}, \quad \hat{\xi}_\alpha = \left\{\mathbf{n}, \sqrt[3]{\frac{\omega}{\rho}}\right\} \tag{69}$$

Evidently, $\hat{\boldsymbol{\xi}}$ satisfies the conditions for fictitious basic internal variables. Since the fictitious cracks are identical with each other, the damage surface of the damage vector, $\mathscr{F}$, is the same as that of each fictitious crack, i.e., $\mathscr{F} = F_\alpha$ and $\dot{\lambda} = \dot{\lambda}_\alpha$. Substitute eqns (67) and (69) into eqn (65)

$$Q(\boldsymbol{\xi}, \mathbf{g}, \mathscr{F}) = \frac{\dot{\lambda}}{2\rho}\left[\mathbf{g}_{\mathrm{n}} \cdot (\mathbf{I} - \mathbf{nn}) \cdot \mathbf{g}_{\mathrm{n}} + \left(\frac{3\omega g_\omega}{2}\right)^2\right] \tag{70}$$

With eqn (19), the kinetic equation of the damage vector is

$$\dot{\zeta} = \{\dot{\mathbf{n}}, \dot{\omega}\} = \frac{\partial Q(\boldsymbol{\zeta}, \mathbf{g}, \mathscr{F})}{\partial g} = \frac{\dot{\lambda}}{\rho}\left\{(\mathbf{I} - \mathbf{nn}) \cdot \mathbf{g}_{\mathrm{n}}, \left(\frac{3\omega}{2}\right)^2 g_\omega\right\} \tag{71}$$

Similarly, the damage surface $\mathscr{F}$ is obtained by substituting eqns (67) and (69) into eqn (66) and with a minor modification $\mathscr{F} = (\pi r_\alpha^2 \rho/\sqrt{2}) F_\alpha$,

$$\mathscr{F} = \mathscr{G} - \mathscr{R}, \quad \mathscr{G}^2 = \frac{1}{2}\left[\mathbf{g}_{\mathrm{n}} \cdot (\mathbf{I} - \mathbf{nn}) \cdot \mathbf{g}_{\mathrm{n}} + \left(\frac{3\omega g_\omega}{2}\right)^2\right] \tag{72}$$

The kinetic equation eqn (71) can be recast as

$$\begin{Bmatrix}\dot{\mathbf{n}} \\ \dot{\omega}\end{Bmatrix} = \frac{\dot{\lambda}}{\rho}\mathbf{L}\begin{Bmatrix}\mathbf{g}_{\mathrm{n}} \\ g_\omega\end{Bmatrix}, \quad \mathbf{L} = \begin{bmatrix}\mathbf{I} - \mathbf{nn} & 0 \\ 0 & \left(\frac{3\omega}{2}\right)^2\end{bmatrix} \tag{73}$$

Evidently, $\mathbf{L}$ satisfies the positive definite condition and Onsager reciprocal relations, see e.g. Malvern (1969). By introducing $\mathbf{L}$, Q and $\mathscr{G}$ have the compact form:

$$Q = \frac{\dot{\lambda}}{2\rho} g \cdot \mathbf{L} \cdot g, \quad \mathscr{G}^2 = \frac{1}{2} g \cdot \mathbf{L} \cdot g \tag{74}$$

7. Kinetic equations of damage tensor

The procedures for obtaining the kinetic equation of the damage tensor $\mathbf{\Omega}$ defined in eqn (5) are parallel to these for the damage vectors $\boldsymbol{\zeta}$. Note that here the damage vectors $\boldsymbol{\zeta}$ should be considered as the basic internal variables. For the full set of damage vectors defined in eqn (3), the general potential function is

$$Q(\boldsymbol{\zeta}, \mathbf{g}, \mathscr{F}) = \sum_{\mu=1}^{m} \frac{\dot{\lambda}_\mu}{2\rho_\mu} g_\mu \cdot \mathbf{L}_\mu \cdot g_\mu \tag{75}$$

and the damage surface $\mathscr{F}$ is one set of the following damage surfaces

$$\mathscr{F}_\mu = \mathscr{G}_\mu - \mathscr{R}_\mu, \quad \mathscr{G}_\mu^2 = \tfrac{1}{2} g_\mu \cdot \mathbf{L}_\mu \cdot g_\mu \quad (\mu = 1, 2, \ldots, m) \tag{76}$$

Now consider the transformation $(\mathbf{g}, \boldsymbol{\zeta}, \mathscr{F}) \Rightarrow (\mathbf{Y}, \mathbf{\Omega}, \hat{\mathscr{F}})$. In view of eqns (68) and (5), $\mathbf{g} = \mathbf{g}(\mathbf{Y})$ is achieved

$$\begin{aligned} g_\mu = \{g_{\mathrm{n}}^\mu, g_\omega^\mu\} &= \left\{\frac{\partial \psi}{\partial \mathbf{n}^\mu}, \frac{\partial \psi}{\partial \omega_\mu}\right\} = \left\{\frac{\partial \psi}{\partial \mathbf{\Omega}} : \frac{\partial \mathbf{\Omega}}{\partial \mathbf{n}^\mu}, \frac{\partial \psi}{\partial \mathbf{\Omega}} : \frac{\partial \mathbf{\Omega}}{\partial \omega_\mu}\right\} \\ &= \mathbf{Y} : \{\omega_\mu (\mathbf{I}\mathbf{n}^\mu + \mathbf{n}^\mu \mathbf{I}), \mathbf{n}^\mu \mathbf{n}^\mu\} \quad \text{(no summation for } \mu) \end{aligned} \tag{77}$$

where $\mathbf{Y}$ is the thermodynamic force conjugate to the damage tensor

$$\mathbf{Y} = \frac{\partial \psi}{\partial \mathbf{\Omega}} = -\frac{\partial \phi}{\partial \mathbf{\Omega}} \tag{78}$$

In order to determine $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\mathbf{\Omega})$, the inverse relation of eqn (5), three characteristic damage vectors $\hat{\boldsymbol{\zeta}}$ are introduced as the fictitious internal variables to replace $\boldsymbol{\zeta}$

$$\hat{\boldsymbol{\zeta}} = \{\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3\}, \quad \hat{\zeta}_v = \{\mathbf{n}^v, \omega_v\} \Leftrightarrow \mathbf{\Omega} = \sum_{v=1}^{3} \omega_v \mathbf{n}^v \mathbf{n}^v \tag{79}$$

where $\mathbf{n}^v$ and ω_v are the principal directions and values of the damage tensor. For $\hat{\boldsymbol{\zeta}}$, all volumetric densities ρ_v are the same, i.e., $\rho_1 = \rho_2 = \rho_3 = \rho$. In terms of the characteristic damage vectors, the damage potential becomes

$$Q(\hat{\boldsymbol{\zeta}}, \mathbf{g}, \hat{\mathscr{F}}) = \sum_{v=1}^{3} \frac{\dot{\lambda}_v}{2\rho} g_v \cdot \mathbf{L}_v \cdot g_v \tag{80}$$

and the damage surfaces $\hat{\mathscr{F}}$ become

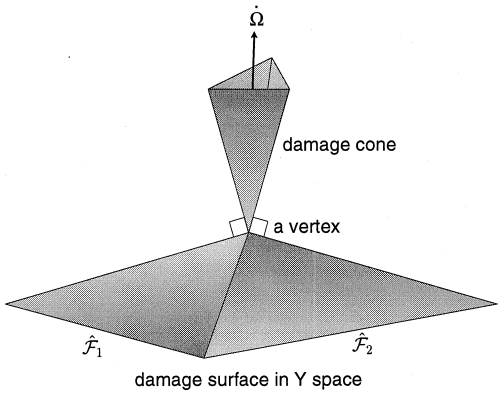

Fig. 6. Schematic vertex structure in the damage surface $\hat{\mathscr{F}}$.

$$\hat{\mathscr{F}}_v = \mathscr{G}_v - \mathscr{R}_v, \quad \mathscr{G}_v^2 = \tfrac{1}{2} g_v \cdot \mathbf{L}_v \cdot g_v \quad (v = 1, 2, 3) \tag{81}$$

7.1. *Vertex-like kinetic equations*

Using eqns (77) and (73), $\mathscr{G}_v^2$ can be recast as

$$\mathscr{G}_v^2 = \tfrac{1}{2} g_v \cdot \mathbf{L}_v \cdot g_v = \tfrac{1}{2} \mathbf{Y} : \mathbf{J}_v : \mathbf{Y} \quad \text{(no summation for } v\text{)} \tag{82}$$

where[3]

$$\mathbf{J}_v = \omega_v^2 (\mathbf{In} + \mathbf{nI}) \cdot (\mathbf{I} - \mathbf{nn}) \cdot (\mathbf{In} + \mathbf{nI}) + \left(\frac{3\omega_v}{2}\right)^2 \mathbf{nnnn} = \omega_v^2 \left(4\mathbf{T}_v + \frac{9}{4}\mathbf{N}_v\right)$$

$$\text{(no summation for } v\text{)} \tag{83}$$

where

$$\mathbf{N}_v = \mathbf{nnnn}, \quad \mathbf{T}_v = T^v_{ijkl} = \tfrac{1}{2}(n_i n_k \delta_{jl} + n_i n_l \delta_{jk} + n_j n_k \delta_{il} + n_j n_l \delta_{ik}) - n_i n_j n_k n_l \tag{84}$$

Then the damage surface of $\hat{\boldsymbol{\zeta}}$ defined in eqn (81) can be recast as

$$\hat{\mathscr{F}}_v = \mathscr{G}_v - \mathscr{R}_v, \quad \mathscr{G}_v^2 = \tfrac{1}{2} \mathbf{Y} : \mathbf{J}_v : \mathbf{Y} \quad (v = 1, 2, 3) \tag{85}$$

The overall damage surface of the damage tensor, $\hat{\mathscr{F}}$, is the inner envelope of the damage surfaces of $\hat{\mathscr{F}}_1$, $\hat{\mathscr{F}}_2$ and $\hat{\mathscr{F}}_3$ defined in eqn (85). Obviously we can expect some pointed vertices on $\hat{\mathscr{F}}$. One of them is shown in Fig. 6.

In view of eqns (80) and (82) and incorporating ρ into λ_v, the damage potential of the damage tensor is obtained as

$$Q(\boldsymbol{\Omega}, \mathbf{Y}, \hat{\mathscr{F}}) = \tfrac{1}{2} \mathbf{Y} : (\lambda_v \mathbf{J}_v) : \mathbf{Y} \tag{86}$$

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[3] To make a compact formulation, the superscript v in $\mathbf{n}^v$ is dropped in eqns (83) and (84).

where the multipliers $\dot{\lambda}_v$ are associated with the damage surfaces $\hat{\mathscr{F}}$. Noting the symmetry of $\mathbf{J}_v$, the kinetic equation is obtained as

$$\dot{\mathbf{\Omega}} = \frac{\partial Q(\mathbf{\Omega}, \mathbf{Y}, \hat{\mathscr{F}})}{\partial \mathbf{Y}} = (\dot{\lambda}_v, \mathbf{J}_v): \mathbf{Y} = (\dot{\lambda}_1 \mathbf{J}_1 + \dot{\lambda}_2 \mathbf{J}_2 + \dot{\lambda}_3 \mathbf{J}_3): \mathbf{Y} \tag{87}$$

which can be illustrated geometrically in $\mathbf{Y}$ space by Hill's (1967) essential structure.[4] It indicates that the direction of $\dot{\mathbf{\Omega}}$ is uncertain but confined in the damage cone, which is normal to the damage surface at a vertex, as shown in Fig. 6.

7.2. *Smooth kinetic equations*

The vertex-like damage surface is associated with an uncertain kinetic equation. The phenomenological eqn (1) exists only if there exists a corresponding smooth damage surface. In this subsection, the smooth damage surface is pursued by a parallel way to plasticity.

The tensors $\mathbf{T}_v$ and $\mathbf{N}_v$ are associated with shear and normal components of a second-order tensor, respectively, on the surface with the normal vector $\mathbf{n}^v$. For example, the normal stress σ_v and shear stress τ_v on the surface are $\sigma_v^2 = \boldsymbol{\sigma}: \mathbf{N}_v: \boldsymbol{\sigma}$ and $\tau_v^2 = \boldsymbol{\sigma}: \mathbf{T}_v: \boldsymbol{\sigma}$ where $\boldsymbol{\sigma}$ is the stress tensor. Similarly, $(Y_v^\sigma)^2 = \mathbf{Y}: \mathbf{N}_v: \mathbf{Y}$ and $(Y_v^\tau)^2 = \mathbf{Y}: \mathbf{T}_v: \mathbf{Y}$ define the normal and shear components of the conjugate force $\mathbf{Y}$, respectively, on the surface. In view of eqns (82) and (83), the energy release rate of the damage vector can be recast as

$$\mathscr{G}_v = \sqrt{2}\omega_v Y_v, \quad Y_v = \sqrt{(Y_v^\sigma)^2 + (\tfrac{3}{4} Y_v^\tau)^2}, \quad \text{(no summation for } v\text{)} \tag{88}$$

Essentially, Y_v is the resultant force[5] of the conjugate force $\mathbf{Y}$ on the surface.

According to eqn (85), the overall damage surface of the damage tensor, $\hat{\mathscr{F}}$, can be recast as the following condition:

$$\hat{\mathscr{F}}: \quad \mathscr{G}_1 \leqslant \mathscr{R}_1, \quad \mathscr{G}_2 \leqslant \mathscr{R}_2, \quad \mathscr{G}_3 \leqslant \mathscr{R}_3 \tag{89}$$

which is similar to the Tresca yield condition (see e.g. Malvern, 1969) which is also vertex-like. Note that the Mises yield condition can be considered as the smoothed version of the Tresca yield condition. In view of the parallel relations between the damage surfaces and the yield conditions, see Table 2, the way of smoothing the Tresca yield condition into the Mises yield condition, can be performed directly to smooth $\hat{\mathscr{F}}$ into a smooth overall damage surface $\tilde{\mathscr{F}}$

$$\tilde{\mathscr{F}} = \tilde{\mathscr{G}} - \tilde{\mathscr{R}}, \quad \tilde{\mathscr{G}}^2 = \mathscr{G}_1^2 + \mathscr{G}_2^2 + \mathscr{G}_3^2 \tag{90}$$

where the general energy release rate $\mathscr{G}$ is, due to eqns (82) and (83),

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[4] Although Hill's theory is concerned with the elasto-plastic behaviours at the stress–strain level, it can be used here directly, because the inelastic strain $\boldsymbol{\varepsilon}^d$ defined in eqn (13) epitomizes plastic and damaging strains, and eqn (19) indicates that $\dot{\zeta}_\mu$, g_μ and $\boldsymbol{\varepsilon}^d$, $\boldsymbol{\sigma}$ have precisely parallel relations.

[5] The coefficient 3 in eqn (88) originates from the order of the moment of radii in eqn (4). If the order is 4, Y_v is exactly the resultant force.

Table 2
Comparison of yield and damage surfaces

	Plasticity	Damage
DR (Drive force)	$\boldsymbol{\sigma}$	$\mathbf{Y}$
DR's normal and tangent components on a surface	$\sigma_v^2 = \boldsymbol{\sigma}:\mathbf{N}_v:\boldsymbol{\sigma}$	$(Y_v^\sigma)^2 = \mathbf{Y}:\mathbf{N}_v:\mathbf{Y}$
	$\tau_v^2 = \boldsymbol{\sigma}:\mathbf{T}_v:\boldsymbol{\sigma}$	$(Y_v^\tau)^2 = \mathbf{Y}:\mathbf{T}_v:\mathbf{Y}$
Effective DR on the surface	τ_v	$Y_v = \sqrt{(Y_v^\sigma)^2 + (\frac{3}{4}Y_v^\tau)^2}$
The characteristic acting surfaces of τ_v, Y_v	Three orthogonal pure shear planes of stress tensor† $\boldsymbol{\sigma}$	Three orthogonal principal planes of damage tensor $\boldsymbol{\Omega}$
Yield/damage surface	$\tau_1 \leqslant [\tau]$, $\tau_2 \leqslant [\tau]$, $\tau_3 \leqslant [\tau]$ (the Tresca condition)	$\mathscr{G}_1 \leqslant \mathscr{R}_1$, $\mathscr{G}_2 \leqslant \mathscr{R}_2$, $\mathscr{G}_3 \leqslant \mathscr{R}_3$ ($\hat{\mathscr{F}}$; where $\mathscr{G}_v = \sqrt{2\omega_v}\,Y_v$)
Smoothed surface	$\sqrt{\tau_1^2+\tau_2^2+\tau_3^2} \leqslant [\tilde{\tau}]$ (the Mises condition)	$\sqrt{\mathscr{G}_1^2+\mathscr{G}_2^2+\mathscr{G}_3^2} \leqslant \tilde{\mathscr{R}}$ ($\tilde{\mathscr{F}}$: the smoothed $\tilde{\mathscr{F}}$)

† Since $\sigma_v = 0$ on a pure shear plane, both τ_v and Y_v can be considered as the resultant forces of the drive force $\boldsymbol{\sigma}$ and $\mathbf{Y}$, respectively, on the characteristic acting surfaces.

$$\mathscr{G}^2 = \frac{1}{2}\mathbf{Y}:\mathbf{J}:\mathbf{Y}, \quad \mathbf{J} = \mathbf{J}(\boldsymbol{\Omega}) = \sum_{v=1}^{3}\mathbf{J}_v = \sum_{v=1}^{3}\omega_v^2\left(4\mathbf{T}_v + \frac{9}{4}\mathbf{N}_v\right) \tag{91}$$

For the smooth damage surface $\tilde{\mathscr{F}}$ the associated damage potential is

$$Q(\boldsymbol{\Omega}, \mathbf{Y}, \tilde{\mathscr{F}}) = \tfrac{1}{2}\dot{\lambda}\mathbf{Y}:\mathbf{J}:\mathbf{Y} \tag{92}$$

As compared with eqn (86), the introduction of $\tilde{\mathscr{F}}$ implies $\dot{\lambda}_1 = \dot{\lambda}_2 = \dot{\lambda}_3 = \dot{\lambda}$. Therefore, the kinetic equation of the damage tensor is

$$\dot{\boldsymbol{\Omega}} = \dot{\lambda}\frac{\partial Q(\boldsymbol{\Omega}, \mathbf{Y}, \tilde{\mathscr{F}})}{\partial \mathbf{Y}} = \dot{\lambda}\mathbf{J}:\mathbf{Y} \tag{93}$$

which is exactly the phenomenological eqn (1) pursued in this paper.

8. Conclusion

The essential problem of this paper has been answered through the deduction toward the phenomenological eqn (1). The properties of microcracks, e.g., the shape, orientation, etc., generally present discontinuous distributions in a solid, which leads to vertex-like constitutive equations. The deduction toward the phenomenological eqn (1) are in reality a series of 'smoothing' procedures: (1) describing a microcrack by a vector variable ξ_α; (2) epitomizing a group of parallel microcracks by a damage vector ζ_μ; (3) condensing all damage vectors $\boldsymbol{\zeta}$ into three characteristic damage vectors $\hat{\boldsymbol{\zeta}}$ or the damage tensor $\boldsymbol{\Omega}$; (4) smoothing the vertex-like damage surface of the damage tensor, $\hat{\mathscr{F}}$, into a smooth one $\tilde{\mathscr{F}}$.

The first 'smoothing' step is to describe one real crack by one self-similar growth penny-shaped

crack ξ_α. The kinetic equation of ξ_α is the cornerstone of all the deduction. To a certain extent, it defines the materials which the deduced constitutive equations represent. Its effectiveness to describe microcracking is demonstrated in Section 5. The second and third 'smoothing' steps are asymptotic approaches for one set of condensed internal variables $\boldsymbol{\zeta}$ to describe the thermodynamic system characterized by the numerous basic internal variables $\boldsymbol{\xi}$. Mathematically, it is the way to deal with the uncertainty of $\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{\zeta})$. In this paper, an effective smoothing procedure for the uncertainty is proposed. The fourth 'smoothing' step, i.e., smoothing $\hat{\mathscr{F}}$ into $\tilde{\mathscr{F}}$, is required for the existence of the phenomenological eqn (1). The specific form of $\tilde{\mathscr{F}}$ defined in eqn (90) is achieved by a parallel way to plasticity. In terms of the driving forces and the smoothing procedures, $\hat{\mathscr{F}}$ is to $\tilde{\mathscr{F}}$ what the Tresca yield condition is to the Mises yield condition. These smoothing procedures are exactly in the spirit of continuum damage mechanics.

As another result of such smoothing, the damage characteristic tensor $\mathbf{J}$, defined in eqns (91) and (84), can be determined uniquely using the current damage tensor without resorting to microscopic parameters or unknown empirical coefficients. $\mathbf{J}$ is a positive-definite anisotropic tensor. It is not adequate to approximate $\mathbf{J}$ by an isotropic tensor. As shown in eqn (93), $\dot{\boldsymbol{\Omega}}$ is not coaxial with $\mathbf{J}$ (in the sense of principal axes) unless $\boldsymbol{\Omega}$ is isotropic or $\mathbf{J}$ is coaxial with $\boldsymbol{\Omega}$. The damage characteristic tensor $\mathbf{J}$ represents the influence of the existing anisotropic microstructure on the damage propagation.

The damage variable takes the form of the third-order moment of microcrack radii in the sense of eqn (4b) and the damage tensor is just the second-order fabric tensor. From a purely statistical viewpoint, the order is not necessary to be three. As pointed out in footnote 5, the order has minor influence on the deduced formulation; the fourth-order moment has a specific mechanical meaning.

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